Head-on collision of two concentric cylindrical ion acoustic solitary waves

Guoxiang Huang* and Manuel G. Velarde
Instituto Pluridisciplinar, Universidad Complutense, Paseo Juan XXIII, No. 1, Madrid 28040, Spain
(Received 13 April 1995)

The head-on collision of two concentric cylindrical, ion acoustic solitary waves traveling in opposite directions is considered by extending the Poincaré-Lighthill-Kuo method to the cylindrical geometry. The results show that the phase shifts of the solitary waves due to the collision are proportional to $r^{-1/3}$ and depend on their initial positions, where r is the radius measured from the center of the disturbances.

PACS number(s): 52.35.Mw, 47.35.+i, 52.35.Sb

Ion acoustic solitary waves have been extensively studied both theoretically and experimentally, resulting in considerable success in clarifying many aspects of the characteristics of solitary waves in planar [1–3], cylindrical, and spherical systems [4–7]. For the interaction of the ion acoustic solitary waves, a series of theoretical and experimental investigations also have been made. In the planar case, theoretical predictions [8,9] have been confirmed by experiments [10–12]. However, for cylindrical and spherical solitary waves only the colliding behavior of nonconcentric solitary waves have been considered [13–16]. There is no detailed study about the head-on collision between two *concentric* cylindrical or spherical solitary waves [17].

In this paper, we investigate the head-on collision between two concentric cylindrical ion acoustic solitary waves based on an extended Poincaré-Lighthill-Kuo (PLK) method.

We consider an ion acoustic wave, that is, a fluctuation in the ion density of a two-component, collisionless plasma. The ions carry low-frequency density and velocity fluctuations near the ion plasma frequency, while the electrons preserve an approximate local charge neutrality by following the ion motion. Higher-frequency fluctuations near the electron plasma frequency will be ignored. In dimensionless form the equations of motion are

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{v})=0\ , \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla\phi\ , \quad (2)$$

$$\nabla^2\phi=\exp(\phi)-n\ , \quad (3)$$

where nn_0 is the ion number density, $\lambda_D\mathbf{x}$ the space coordinate, $\omega_0 t$ the time, $(\omega_0\lambda_D)\mathbf{v}$ the ion velocity, and $(k_B T_e/e)\phi$ the electrostatic potential. Here n_0 is the equilibrium value of the ion number density, λ_D the Debye shielding length of the electron, ω_0 the ion plasma frequency, T_e the electron temperature, $-e$ the electron charge, and k_B the Boltzmann constant.

In cylindrical coordinates, Eqs. (1)–(3) have the form

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rnv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(nv_\theta)+\frac{\partial}{\partial z}(nv_z)=0\ , \quad (4)$$

$$\frac{dv_r}{dt}-\frac{1}{r}v_\theta^2+\frac{\partial\phi}{\partial r}=0\ , \quad (5)$$

$$\frac{dv_\theta}{dt}+\frac{1}{r}v_r v_\theta+\frac{1}{r}\frac{\partial\phi}{\partial\theta}=0\ , \quad (6)$$

$$\frac{dv_z}{dt}+\frac{\partial\phi}{\partial z}=0\ , \quad (7)$$

$$\nabla^2\phi=\exp(\phi)-n\ , \quad (8)$$

where r, θ, and z are the radial, polar angle, and vertical coordinate, respectively. v_r, v_θ, and v_z represent the components of the ion velocity in the cylindrical coordinate system. ∇^2 is the Laplacian operator. In the following, for simplicity, we only consider the excitations with cylindrical symmetry and independent of z as in Ref. [4].

Suppose that two cylindrical solitary waves, R and L, have been excited in the system. The solitary wave R (L) is traveling outward (inward) from (to) the initial point of the coordinate system. The initial position (at time $t=0$) of the solitary wave R (L) is at $r=r_R$ ($r=r_L$), $r_L \gg r_R$ [18]. After some time they interact, following a collision, and then depart each other. In order to investigate the head-on collision between the two cylindrical solitary waves, we extend the PLK method [19–22] to the cylindrical geometry. We anticipate that the collision will result in phase shifts in their postcollision trajectories. Thus we introduce the following transformation:

$$\xi=\epsilon(r-t-r_R)+\epsilon^2P_0(\eta,\tau)+\epsilon^3P_1(\xi,\eta,\tau)+\cdots\ , \quad (9)$$

$$\eta=\epsilon(r+t-r_L)+\epsilon^2Q_0(\xi,\tau)+\epsilon^3Q_1(\xi,\eta,\tau)+\cdots\ , \quad (10)$$

$$\tau=\epsilon^3 r\ , \quad (11)$$

where ϵ is the smallness and ordering parameter for the series expansion and P_j and Q_j ($j=0,1,2,\ldots$) are the functions to be determined in the process of our perturbation solution of (4)–(8). Thus for the spatial and temporal derivatives we have

$$\frac{\partial}{\partial r}=\epsilon\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]+\epsilon^3\left[\frac{\partial}{\partial\tau}+P_{0\eta}\frac{\partial}{\partial\xi}+Q_{0\xi}\frac{\partial}{\partial\eta}\right]+\cdots\ , \quad (12)$$

*On leave from Department of Physics, East China Normal University, Shanghai 200062, People's Republic of China.

$$\frac{\partial}{\partial t}=\epsilon\left[-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]+\epsilon^3\left[P_{0\eta}\frac{\partial}{\partial\xi}-Q_{0\xi}\frac{\partial}{\partial\eta}\right]+\cdots\ . \tag{13}$$

Introducing the asymptotic expansion

$$v_r=\epsilon^2(u_0+eu_1+\epsilon^2u_2+\cdots)\ , \tag{14}$$

$$n=1+\epsilon^2(n_0+\epsilon n_1+\epsilon^2 n_2+\cdots)\ , \tag{15}$$

$$\phi=\epsilon^2(\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\cdots)\ , \tag{16}$$

and substituting (12)–(16) into (4), (5), and (8), one obtains a hierarchy of linear, inhomogeneous equations:

$$\left[-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]n_j+\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]u_j=\alpha_j\ , \tag{17}$$

with

$$\alpha_0=\alpha_1=0\ , \tag{18}$$

$$\alpha_2=-\left[\frac{1}{\tau}+\frac{\partial}{\partial\tau}\right]u_0-\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right](n_0u_0)-\left[P_{0\eta}\frac{\partial}{\partial\xi}-Q_{0\xi}\frac{\partial}{\partial\eta}\right]n_0-\left[P_{0\eta}\frac{\partial}{\partial\xi}+Q_{0\xi}\frac{\partial}{\partial\eta}\right]u_0\ , \tag{19}$$

...

$$\left[-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]u_j+\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]\phi_j=\beta_j\ , \tag{20}$$

with

$$\beta_0=\beta_1=0\ , \tag{21}$$

$$\beta_2=-u_0\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]u_0-\left[P_{0\eta}\frac{\partial}{\partial\xi}-Q_{0\xi}\frac{\partial}{\partial\eta}\right]u_0-\left[\frac{\partial}{\partial\tau}+P_{0\eta}\frac{\partial}{\partial\xi}+Q_{0\xi}\frac{\partial}{\partial\eta}\right]\phi_0\ , \tag{22}$$

...

and

$$n_j-\phi_j=\gamma_j\ , \tag{23}$$

$$\gamma_0=\gamma_1=0\ , \tag{24}$$

$$\gamma_2=-\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]^2\phi_0+\frac{1}{2}\phi_0^2\ , \tag{25}$$

...

with $j=0,1,2,\ldots$. By (17), (20), and (23), we obtain an equation for u_j:

$$4\frac{\partial^2}{\partial\xi\partial\eta}u_j=\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]\alpha_j+\left[\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right]\beta_j+\left[\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}\right]\gamma_j\ , \tag{26}$$

$j=0,1,2,\ldots$.

At the leading order ($j=0$), we obtain the solution

$$u_0=f_0(\xi,\tau)-q_0(\eta,\tau)\ , \tag{27}$$

$$n_0=f_0(\xi,\tau)+q_0(\eta,\tau)\ , \tag{28}$$

$$\phi_0=f_0(\xi,\tau)+q_0(\eta,\tau)\ , \tag{29}$$

where f_0 and q_0 are to be determined. Hence in the leading order we have two waves, one of which, $f_0(\xi,\tau)$, travels outward, and the other one, $q_0(\eta,\tau)$, inward.

To the order $O(\epsilon)$ ($j=1$), the solution is

$$u_1=f_1-q_1\ ,\quad n_0=\phi_1=f_1+q_1\ , \tag{30}$$

where $f_1(\xi,\tau)$ and $q_1(\eta,\tau)$ are to be determined.

To order $O(\epsilon^2)$ ($j=2$), Eq. (26) reads

$$4\frac{\partial^2}{\partial\xi\partial\eta}u_2=-\frac{\partial}{\partial\xi}\left[\left[2\frac{\partial}{\partial\tau}+\frac{1}{\tau}\right]f_0+2f_0f_{0\xi}+f_{0\xi\xi\xi}\right]+\frac{\partial}{\partial\eta}\left[\left[2\frac{\partial}{\partial\tau}+\frac{1}{\tau}\right]q_0+2q_0q_{0\eta}+q_{0\eta\eta\eta}\right]-2[(2P_{0\eta}-q_0)f_{0\xi\xi}+(2Q_{0\xi}-f_0)q_{0\eta\eta}]\ , \tag{31}$$

or

$$u_2=-\frac{1}{4}\int d\eta\left[\left[2\frac{\partial}{\partial\tau}+\frac{1}{\tau}\right]f_0+2f_0f_{0\xi}+f_{0\xi\xi\xi}\right]+\frac{1}{4}\int d\xi\left[\left[2\frac{\partial}{\partial\tau}+\frac{1}{\tau}\right]q_0+2q_0q_{0\eta}+q_{0\eta\eta\eta}\right]-\frac{1}{2}\int\int d\xi d\eta[(2P_{0\eta}-q_0)f_{0\xi\xi}]-\frac{1}{2}\int\int d\xi\, d\eta[(2Q_{0\xi}-f_0)q_{0\eta\eta}]\ . \tag{32}$$

The first (second) term of (32) will be proportional to $\eta(\xi)$ because the integrated function is independent of η (ξ). Thus the first two terms of (32) are all *secular* terms, which must be eliminated in order to avoid spurious resonances [20–22]. Hence we have:

$$\frac{\partial f_0}{\partial\tau}+\frac{1}{2\tau}f_0+f_0\frac{\partial f_0}{\partial\xi}+\frac{1}{2}\frac{\partial^3 f_0}{\partial\xi^3}=0\ , \tag{33}$$

$$\frac{\partial q_0}{\partial\tau}+\frac{1}{2\tau}q_0+q_0\frac{\partial q_0}{\partial\eta}+\frac{1}{2}\frac{\partial^3 q_0}{\partial\eta^3}=0\ . \tag{34}$$

The third and the forth terms in (32) are not secular terms in this order, but they will become secular in the next orders [20–22]. Hence we have

$$P_{0\eta}=\tfrac{1}{2}q_0\ , \tag{35}$$

$$Q_{0\xi}=\tfrac{1}{2}f_0\ . \tag{36}$$

Equations (33) and (34) are cylindrical Korteweg–de Vries equations. Their single- or multisoliton solutions can be obtained by the inverse scattering transform [23] or by Hirota's method [24]. Equations (35) and (36) permit calculating the leading phase changes in the head-on collision.

Since we are interested in the asymptotic properties of the collision process, we use the asymptotic solutions of (33) and (34), rather than their exact solutions, which involve the Airy function [24]. By using the method given by Kako and Yajima [14] or Weidman and Zakhem [25], for large τ we obtain the quasisolitary wave solutions of (33) and (34):

$$f_0 = A_R \left[\frac{\tau_R}{\tau}\right]^{2/3} \operatorname{sech}^2\left\{\left[\frac{A_R}{6}\right]^{1/2}\left[\frac{\tau_R}{\tau}\right]^{1/3}\left[\xi - A_R\left[\frac{\tau_R}{\tau}\right]^{2/3}\tau\right]\right\}, \tag{37}$$

$$q_0 = A_L \left[\frac{\tau_L}{\tau}\right]^{2/3} \operatorname{sech}^2\left\{\left[\frac{A_L}{6}\right]^{1/2}\left[\frac{\tau_L}{\tau}\right]^{1/3}\left[\xi - A_L\left[\frac{\tau_L}{\tau}\right]^{2/3}\tau\right]\right\}, \tag{38}$$

where $\tau_R = \epsilon^3 r_R$ and $\tau_L = \epsilon^3 r_L$. A_R (A_L) is the amplitude of the cylindrical solitary wave R (L) at the initial position $r = r_R$ ($r = r_L$). From (37) and (38) we can calculate the leading phase changes due to the collision. We have

$$\begin{aligned} P_0 &= \frac{1}{2}\int_{\eta|_{t=0}}^{\eta} q_0(\eta',\tau)\,d\eta' \\ &= (\tfrac{3}{2}A_L)^{1/2}\left[\frac{\tau_L}{\tau}\right]^{1/3}\left[\tanh\left\{\left[\frac{A_L}{6}\right]^{1/2}\left[\frac{\tau_L}{\tau}\right]^{1/2}\left[\eta - A_L\left[\frac{\tau_L}{\tau}\right]^{2/3}\tau\right]\right\}\right. \\ &\qquad\left. -\tanh\left\{\left[\frac{A_L}{6}\right]^{1/2}\left[\frac{\tau_L}{\tau}\right]^{1/3}\left[\eta|_{t=0} - A_L\left[\frac{\tau_L}{\tau}\right]^{2/3}\tau\right]\right\}\right], \end{aligned} \tag{39}$$

$$\begin{aligned} Q_0 &= \frac{1}{2}\int_{\xi|_{t=0}}^{\xi} f_0(\xi',\tau)\,d\xi' \\ &= \left[\frac{3}{2}A_R\right]^{1/2}\left[\frac{\tau_L}{\tau}\right]^{1/3}\left[\tanh\left\{\left[\frac{A_R}{6}\right]^{1/2}\left[\frac{\tau_R}{\tau}\right]^{1/3}\left[\xi - A_R\left[\frac{\tau_R}{\tau}\right]^{2/3}\tau\right]\right\}\right. \\ &\qquad\left. -\tanh\left\{\left[\frac{A_R}{6}\right]^{1/2}\left[\frac{\tau_R}{\tau}\right]^{1/3}\left[\xi|_{t=0} - A_R\left[\frac{\tau_R}{\tau}\right]^{2/3}\tau\right]\right\}\right], \end{aligned} \tag{40}$$

where $\xi|_{t=0} = -\eta|_{t=0} = \epsilon(r_L - r_R)$. Then we obtain the solution up to $O(\epsilon^2)$.

We can estimate the phase shifts in the head-on collision process of the two concentric cylindrical solitary waves traveling in opposite directions. The phase shift Δ_R (Δ_L) for solitary wave R (L) is

$$\begin{aligned} \Delta_R = -\epsilon^2(\tfrac{3}{2}A_L)^{1/2}\left[\frac{r_L}{r}\right]^{1/3}&\left[\tanh\left\{\left[\frac{A_L}{6}\right]^{1/2}\left[\frac{r_L}{r}\right]^{1/3}\left[\epsilon(2t + r_R - R_L) - \epsilon^3 A_L\left[\frac{r_L}{r}\right]^{2/3} r\right]\right\}\right. \\ &\left. -\tanh\left\{\left[\frac{A_L}{6}\right]^{1/2}\left[\frac{r_L}{r}\right]^{1/3}\left[\epsilon(r_R - r_L) - \epsilon^3 A_L\left[\frac{r_L}{r}\right]^{2/3} r\right]\right\}\right], \end{aligned} \tag{41}$$

$$\begin{aligned} \Delta_L = -\epsilon^2\left[\frac{3}{2}A_R\right]^{1/2}\left[\frac{r_R}{r}\right]^{1/3}&\left[\tanh\left\{\left[\frac{A_R}{6}\right]^{1/2}\left[\frac{r_R}{r}\right]^{1/2}\left[\epsilon(-2t + r_L - r_R) - \epsilon^3 A_R\left[\frac{r_R}{r}\right]^{2/3} r\right]\right\}\right. \\ &\left. -\tanh\left\{\left[\frac{A_R}{6}\right]^{1/2}\left[\frac{r_R}{r}\right]^{1/3}\left[\epsilon(r_L - r_R) - \epsilon^3 A_R\left[\frac{r_R}{r}\right]^{2/3} r\right]\right\}\right], \end{aligned} \tag{42}$$

when returning to the original variables.

The colliding position of the solitary waves is at $r_C = \frac{1}{2}(r_R + r_L)$ and the collision time is $t_C = \frac{1}{2}(r_L - r_R)$. If the initial distance between two solitary waves is large enough, i.e., $r_L - r_R \gg 1$, and the observation time $t \gg t_C = \frac{1}{2}(r_L - r_R)$, from (41) and (42) we get

$$\Delta_R = -\epsilon^2(6A_L)^{1/2}\left[\frac{r_L}{r}\right]^{1/3}, \tag{43}$$

$$\Delta_L = \epsilon^2(6A_R)^{1/2}\left[\frac{r_R}{r}\right]^{1/3}, \tag{44}$$

which satisfy

$$\frac{1}{\sqrt{A_L}}\left[\frac{1}{r_L}\right]^{1/3}\Delta_R + \frac{1}{\sqrt{A_R}}\left[\frac{1}{r_R}\right]^{1/3}\Delta_L = 0\,. \tag{45}$$

Equation (45) is a phase-conserving relation in the collision.

The phase trajectories of the two solitary waves may be obtained by setting $\xi - A_R(\tau_R/\tau)^{2/3}\tau = 0$ (for solitary wave R) and $\eta - A_L(\tau_L/\tau)^{2/3}\tau = 0$ (for solitary wave L). For small ϵ they become

$$t = (r - r_R) - \epsilon^2 A_R\left[\frac{r_R}{r}\right]^{2/3} r + \epsilon^2\frac{1}{2}(r - r_R)A_L\left[\frac{r_L}{r}\right]^{2/3} \tag{46}$$

for solitary wave R, and

$$t = -(r - r_L) + \epsilon^2 A_L\left[\frac{r_L}{r}\right]^{2/3} r + \epsilon^2\frac{1}{2}(r - r_R)A_L\left[\frac{r_L}{r}\right]^{2/3} \tag{47}$$

for solitary wave L.

From (41) and (42) we see that the phase shifts of two concentric cylindrical solitary waves due to the head-on collision are proportional to $r^{-1/3}$ and depend on their initial positions, r_R and r_L. This results is a geometric effect in the cylindrical system, and it is absent for Cartesian solitons. However, the phase-conserving law (45) is a kind of dynamical effect of the collision. For the phase trajectories of the solitary waves, both the geometric effect and the dynamical effect in the same order contribute to the collision. This can be seen from the second and the third terms of (46) and (47). Higher-order corrections, not considered here, may give some secondary structures in the collision event, and postcollision trajectories.

In the limit $r \rightarrow \infty$, the phase shifts experiencing a head-on collision, given by Eqs. (43) and (44), reduce to those of two Cartesian ion acoustic solitary waves. It suffices to extend the region of radial coordinate from $0<r<+\infty$ to $-\infty<r<+\infty$. Then the initial position for the solitary wave R (L) is $r_R=-\infty$ ($r_L=+\infty$). Thus the phase shifts (46) and (47) become $\Delta_R \rightarrow -\epsilon^2(6A_L)^{1/2}$ and $\Delta_L \rightarrow \epsilon^2(6A_R)^{1/2}$, identical with the results of Oikawa and Yajima [26] (for water waves see Ref. [21], a similar result is expected).

In conclusion, our study of the head-on collision between two concentric cylindrical ion acoustic solitary waves has shown some new effects, geometric and dynamic, given by (43)–(47), respectively, which are absent for Cartesian solitons. These results, which could be tested in experiments of nonlinear ion acoustic waves in a plasma, have been obtained by extending the PLK method to the cylindrical geometry. Needless to say this extended PLK method can also be applied to the study of the head-on collision between two concentric spherical ion acoustic solitary waves and other physical systems.

The authors acknowledge fruitful discussions with J. Bragard, V. N. Kurdyumov, H. Linde, J. Pontes, A. Ye. Rednikov, and A. Wierschem. This research has been supported by DGICYT (Spain) under Grant No. 93-081 and by the Fundación "Ramón Areces" (Spain).

[1] E. Infeld and G. Rowlands, *Nonlinear Waves, Solitons and Chaos* (Cambridge Univ. Pres, Cambridge, 1992).
[2] H. Washimi and T. Taniuti, Phys. Rev. Lett. **17**, 996 (1966).
[3] H. Ikezi, R. J. Taylor, and D. R. Baker, Phys. Rev. Lett. **25**, 11 (1970).
[4] S. Maxon and J. Viecelli, Phys. Fluids **17**, 1614 (1974); Phys. Rev. Lett. **32**, 4 (1974).
[5] N. Hershkowitz and T. Romesser, Phys. Rev. Lett. **32**, 581 (1974).
[6] K. Ko and H. H. Kuehl, Phys. Fluids **22**, 1343 (1979).
[7] Y. Nakamura, M. Ooyama, and T. Ogino, Phys. Rev. Lett. **45**, 1965 (1980).
[8] N. Yajima, M. Oikawa, and J. Satsuma, J. Phys. Soc. Jpn. **44**, 1711 (1978).
[9] F. Kako and N. Yajima, J. Phys. Soc. Jpn. **49**, 2062 (1980).
[10] P. A. Folkes, H. Ikezi, and R. Davies, Phys. Rev. Lett. **45**, 902 (1980).
[11] Y. Nishida and T. Nagasawa, Phys. Rev. Lett. **45**, 1626 (1980).
[12] T. Nagasawa and Y. Nishida, Phys. Rev. A **28**, 3043 (1983).
[13] D. J. Kaup, Physica **2D**, 389 (1981).
[14] F. Kako and N. Yajima, J. Phys. Soc. Jpn. **51**, 311 (1982).
[15] M. Khazel, J. M. Bulson, and K. E. Lonngren, Phys. Fluids **25**, 759 (1982).
[16] I. Tsukabayashi, Y. Nakamura, F. Kako, and K. E. Lonngren, Phys. Fluids **26**, 790 (1983).
[17] In the study provided in Ref. [1] of head-on collisions between two concentric cylindrical and spherical ion acoustic solitary waves, the authors only derived Boussinesq-like equations in cylindrical and spherical geometries under the assumption of potential flow (see Appendix 4 and some comments in pages 303 and 324 of Ref. [1]). Nothing is said about the phase shifts due to the collision.
[18] Since $0<r<\infty$, this consideration is different from Cartesian solitons, where $-\infty<x<\infty$ in one dimension. For the head-on collision between two Cartesian solitons, one can consider one soliton is at $x \approx -\infty$ and the other one at $x \approx \infty$ at the initial time $t=-\infty$ (see Refs. [18–21]).
[19] D. J. Benney and J. C. Luke, J. Math. Phys. **43**, 4 (1964).
[20] M. Oikawa and N. Yajima, J. Phys. Soc. Jpn. **34**, 1093 (1973).
[21] C. H. Su and R. M. Mirie, J. Fluid Mech. **98**, 509 (1980).
[22] Guoxiang Huang, Senyue Lou, and Zaixin Xu, Phys. Rev. E **47**, R3830 (1993).
[23] F. Calogero and A. Degasperis, Lett. Nuovo Cimento **23**, 150 (1978).
[24] A. Nakamura, J. Phys. Soc. Jpn. **49**, 2380 (1980).
[25] P. D. Weidman and R. Zakhem, J. Fluid Mech. **191**, 557 (1988).
[26] See page 1098 of Ref. [20] (noticing that the expansion parameter ϵ used in the present paper corresponds to $\epsilon^{1/2}$ of Ref. [20]).